When first writing Boolean algebra proofs, it is common to try to "use" subtraction, **SCORE: _____ / 15 PTS**
except that subtraction doesn't exist in a Boolean algebra. It is then common to try to define subtraction as adding the complement.
This question shows why that definition also doesn't produce the desired result.

Justify the correct steps of the following INVALID proof by filling in the corresponding blanks, and explain very briefly the error in the incorrect step.

"Given a Boolean algebra B, for all $a, b, c \in B$, if $a+c=b+c$, then $a=b$."

INVALID "PROOF":

Let $a, b, c \in B$ such that $a+c=b+c$.

So, $(a+c)+\bar{c}=(b+c)+\bar{c}$ by substitution.

So, $a+(c+\bar{c})=b+(c+\bar{c})$ by ASSOCIATIVE LAW.

So, $a+1=b+1$ by COMPLEMENT LAW.

So, $a=b$ by ERROR : $a+1=1$, NOT a

Prove the following statement **using an element proof**. **SCORE: _____ / 25 PTS**

For all $A, B, C \subseteq U$, if $A \subseteq B$ and $B \subseteq C^C$, then $A-C=A$.

LET $A, B, C \subseteq U$ SUCH THAT $A \subseteq B$ AND $B \subseteq C^c$

LET $x \in A-C$

SO $x \in A$ AND $x \notin C$ BY DEF'N OF $-$

SO $x \in A$

SO $A-C \subseteq A$ BY DEF'N OF $\subseteq$

LET $x \in A$

SO $x \in B$ BY DEF'N OF $\subseteq$

SO $x \in C^c$ BY DEF'N OF $\subseteq$

SO $x \notin C$ BY DEF'N OF ^{c}

SO $x \in A$ AND $x \notin C$

SO $x \in A-C$ BY DEF'N OF $-$

SO $A \subseteq A-C$ BY DEF'N OF $\subseteq$

SO $A-C=A$ BY DEF'N OF $=$

State the Quotient-Remainder Theorem. **Use correct English.** SCORE: _____ / 6 PTS
You may symbolic logic and set notation, if you use it correctly.
NOTE: You do NOT need to state the definitions of div and mod as part of the theorem.

FOR ALL $n \in \mathbb{Z}$ AND ALL $d \in \mathbb{Z}^+$
THERE EXIST UNIQUE $q, r \in \mathbb{Z}$
SUCH THAT $n = dq + r$ AND $0 \le r < d$

Let $A = \{n \in Z : 3 \mid n\}$ and $B = \{n \in Z : 3 \mid n^2\}$. Prove that $B \subseteq A$. SCORE: _____ / 45 PTS
NOTE: A colon was used for "such that" in set builder notation to make the problem easier to read.

LET $x \in B$
SO $x \in \mathbb{Z}$ AND $3 \mid x^2$
SUPPOSE $x \notin A$
SO $x \notin \mathbb{Z}$ OR $3 \nmid x$
BUT $x \in \mathbb{Z}$, SO $3 \nmid x$
BY EXERCISE 26 IN SECTION 4.4, $x \bmod 3 \neq 0$
BY QRT, $x = 3q+1$ OR $x = 3q+2$ FOR SOME $q \in \mathbb{Z}$
CASE 1 $(x = 3q+1)$: $x^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$
WHERE $3q^2 + 2q \in \mathbb{Z}$ BY CLOSURE OF $\mathbb{Z}$ UNDER $\times$ AND $+$
SO $x^2 \bmod 3 = 1$
SO $3 \nmid x^2$ BY EXERCISE 26 IN SECTION 4.4
SO $x \notin B$
CASE 2 $(x = 3q+2)$: $x^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$
WHERE $3q^2 + 4q + 1 \in \mathbb{Z}$ BY CLOSURE OF $\mathbb{Z}$ UNDER $\times$ AND $+$
SO $x^2 \bmod 3 = 1$
SO $3 \nmid x^2$ BY EXERCISE 26 IN SECTION 4.4
SO $x \notin B$
SO $x \notin B$
BUT $x \in B$ (CONTRADICTION)
SO $x \in A$ BY CONTRADICTION
SO $B \subseteq A$ BY DEF'N OF $\subseteq$

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Let $A = \{x \in U : x \text{ is prime}\}$.
Let $B = \{x \in U : x \bmod 4 = 3\}$.
Let $C = \{x \in U : 3 \mid (x-6)\}$.
Find $A^C - (B \cup C)$.

SCORE: _____ / 12 PTS

NOTE: A colon was used for "such that" in set builder notation to make the problem easier to read.

$A = \{2,3,5,7\}$ $\quad B = \{3,7\}$

$A^c = \{1,4,6,8,9,10\}$ $\quad C = \{3,6,9\}$

$B \cup C = \{3,6,7,9\}$

$A^c - (B \cup C) = \{1,4,8,10\}$

Find and prove an explicit formula for the sequence defined recursively by

SCORE: _____ / 35 PTS

$$a_{n+1} = 2a_n + 5 \text{ for all } n \in Z^+, \quad a_1 = -2$$

$a_2 = 2(-2)+5$

$a_3 = 2(2(-2)+5)+5 = 2\cdot 2\cdot(-2) + 2\cdot 5+5$

$a_4 = 2(2\cdot 2\cdot(-2) + 2\cdot 5+5)+5 = 2\cdot 2\cdot 2\cdot(-2) + 2\cdot 2\cdot 5 + 2\cdot 5+5$

$$a_n = 2^{n-1}(-2) + \frac{5(2^{n-1}-1)}{2-1}$$

$$= -2\cdot 2^{n-1} + 5\cdot 2^{n-1} - 5$$

$$= 3\cdot 2^{n-1} - 5$$

PROOF BY MI:

BASIS STEP: $a_1 = 3\cdot 2^{1-1} - 5 = -2$ ✓

INDUCTIVE STEP: ASSUME $a_k = 3\cdot 2^{k-1} - 5$ FOR SOME P.B.A. $k \in \mathbb{Z}^+$

$$a_{k+1} = 2(3\cdot 2^{k-1} - 5) + 5 = 3\cdot 2^k - 10 + 5$$

$$= 3\cdot 2^k - 5$$

BY MI, $a_n = 3\cdot 2^{n-1} - 5$ FOR ALL $n \in \mathbb{Z}^+$

Two of the following three statements are false. **SCORE: _____ / 12 PTS**

Identify clearly, and provide a counterexample for, each false statement. Show that your counterexample proves the statement is false.

NOTES: **An explanation why a false statement is false is NOT enough. A counterexample is required.**
You do NOT need to prove the true statement is true.

[a] For all $x, y \in Z$, if x and y are both prime, and $x - y > 2$, then $x - y$ is composite.

[b] The sum of two positive irrational numbers is irrational.

[c] For all $x, n \in Z^+$, if $n \geq 2$, $(-x) \bmod n + x \bmod n = n$.

[a] $x = 5$, $y = 2$ ARE BOTH PRIME
$x - y = 3 > 2$ AND 3 IS NOT COMPOSITE

[b] $2 + \sqrt{2}$ AND $2 - \sqrt{2}$ ARE POSITIVE AND IRRATIONAL
$(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4 = \frac{4}{1}$ IS RATIONAL